

Topological Orbit Dimension of MF C*-algebras

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ABSTRACT. This paper is a continuation of our work on D. Voiculescu's topological free entropy dimension $\delta_{\text{top}}(x_1, \dots, x_n)$ for $\vec{x} = (x_1, \dots, x_n)$ of elements in a unital C*-algebra. In this paper we first prove that $\delta_{\text{top}}(x_1, \dots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}$ in which $C^*(\vec{x})$ is MF-nuclear and inner QD. Then we give a relation between the topological orbit dimension $\mathfrak{K}_{\text{top}}^{(2)}$ and the modified free orbit dimension $\mathfrak{K}_2^{(2)}$ by using MF-traces. We also introduce a new invariant $\mathfrak{K}_{\text{top}}^{(3)}$ which is a modification of the topological orbit dimension $\mathfrak{K}_{\text{top}}^{(2)}$ when $\mathfrak{K}_{\text{top}}^{(2)}$ is defined. As the applications of $\mathfrak{K}_{\text{top}}^{(3)}$, We prove that $\mathfrak{K}_{\text{top}}^{(3)}(\mathcal{A}) = 0$ if \mathcal{A} has property $c^*\text{-}\Gamma$ and has no finite-dimensional representations. We also give the definition of property MF- $c^*\text{-}\Gamma$. We then conclude that, for the unital MF C*-algebra $\mathcal{A} = C^*(x_1, x_2, \dots, x_n)$ with no finite-dimensional representations, if \mathcal{A} has property MF- $c^*\text{-}\Gamma$, then $\mathfrak{K}_{\text{top}}^{(3)}(\mathcal{A}) = 0$.

1. Introduction

This paper is a continuation of the work in [11], [15], [18] on D. Voiculescu's topological free entropy dimension $\delta_{\text{top}}(x_1, \dots, x_n)$ for an n -tuple $\vec{x} = (x_1, \dots, x_n)$ of elements in a unital C*-algebra. Here we first give a relation between the topological orbit dimension $\mathfrak{K}_{\text{top}}^{(2)}$ and the modified free orbit dimension $\mathfrak{K}_2^{(2)}$ by using MF-traces. This result allows us to give a new proof of our main result in [18], which gave an estimation of the upper bound of topological free entropy dimension for MF-nuclear algebras. In [18], we have shown that $\delta_{\text{top}}(x_1, \dots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}$ if $C^*(x_1, \dots, x_n)$ is MF-nuclear and residually finite-dimensional (RFD). In [2], Blackadar and Kirchberg proved that all RFD C*-algebras are inner quasidiagonal (inner QD). So it is natural to ask whether the topological free entropy dimension of an MF-nuclear and inner QD algebra is unrelated to its generating family. In this paper we prove that $\delta_{\text{top}}(x_1, \dots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}$ if $C^*(x_1, \dots, x_n)$ is MF-nuclear and inner QD.

In this article, we also introduce a new invariant $\mathfrak{K}_{\text{top}}^{(3)}$ which is a modification of the topological orbit dimension $\mathfrak{K}_{\text{top}}^{(2)}$ when $\mathfrak{K}_{\text{top}}^{(2)}$ is defined. The idea for defining

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$\mathfrak{K}_{top}^{(3)}$ arise from the concept \mathfrak{K}_3 in [12]. We then extend the domain of $\mathfrak{K}_{top}^{(3)}$ to all MF algebras and prove that $\mathfrak{K}_{top}^{(3)}$ is a C^* -algebra invariant. We also modify the notion \mathfrak{K}_3 in [12] by using the modified free orbit dimension $\mathfrak{K}_2^{(2)}$ and denote it by $\mathfrak{K}_3^{(3)}$. We give a relation between $\mathfrak{K}_{top}^{(3)}$ and $\mathfrak{K}_3^{(3)}$ for every MF algebra by using the relation between the topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ and the modified free orbit dimension $\mathfrak{K}_2^{(2)}$. Several properties of $\mathfrak{K}_{top}^{(3)}$ are given as follows:

- (1) $\mathfrak{K}_{top}^{(3)}(\mathcal{N}_1) = \mathfrak{K}_{top}^{(3)}(\mathcal{N}_2)$ if $C^*(\mathcal{N}_1) = C^*(\mathcal{N}_2)$.
- (2) If \mathcal{A} is finite generated, then $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$ if $\mathfrak{K}_{top}^{(2)}(\mathcal{A}) = 0$.
- (3) If $\mathcal{N}_1 \cap \mathcal{N}_2$ has no finite-dimensional representation, then

$$\mathfrak{K}_{top}^{(3)}(C^*(\mathcal{N}_1 \cup \mathcal{N}_2)) \leq \mathfrak{K}_{top}^{(3)}(\mathcal{N}_1) + \mathfrak{K}_{top}^{(3)}(\mathcal{N}_2).$$

- (4) If \mathcal{N} is an MF C^* -algebra and $\mathcal{A} \subseteq \mathcal{N}$ is a C^* -subalgebra with no finite-dimensional representation. If there is an unitary $u \in \mathcal{N}$ such that $u\mathcal{A}u^* \subseteq \mathcal{A}$. Then

$$\mathfrak{K}_{top}^{(3)}(C^*(\mathcal{A} \cup \{u\})) \leq \mathfrak{K}_{top}^{(3)}(\mathcal{A}).$$

As an application, we prove that $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$ if \mathcal{A} has property $c^*\text{-}\Gamma$ and has no finite-dimensional representations. We also give the definition of property MF- $c^*\text{-}\Gamma$. We then conclude that, for the unital MF C^* -algebra $\mathcal{A} = C^*(x_1, x_2, \dots, x_n)$ with no finite-dimensional representations, if \mathcal{A} has property MF- $c^*\text{-}\Gamma$, then $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$.

The organization of the paper is as follows. In section 2, we recall the definition of topological free entropy dimension $\delta_{top}(x_1, \dots, x_n)$ and topological orbit dimension $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n)$ of n -tuple (x_1, \dots, x_n) of elements in a unital C^* -algebra. In section 3, we first give a relation between $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n)$ and $\sup_{\tau \in \mathcal{T}_{MF}(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n, \tau)$ where $\mathcal{T}_{MF}(\mathcal{A})$ is the set of all MF-traces on C^* -algebra $\mathcal{A} = C^*(x_1, \dots, x_n)$. Then we give a new proof of our main result in [18]. In section 4, we discuss the topological free entropy dimension of MF-nuclear and inner QD C^* -algebra, we show that $\delta_{top}(x_1, \dots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}$ as $\mathcal{A} = C^*(x_1, \dots, x_n)$ MF-nuclear and inner QD. We introduce topological orbit dimension $\mathfrak{K}_{top}^{(3)}$ for general MF-algebras in section 5. Several properties of $\mathfrak{K}_{top}^{(3)}$ are discussed there. Section 6 is focus on the applications of $\mathfrak{K}_{top}^{(3)}$ in central sequence algebras. We prove that $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$ if \mathcal{A} has property $c^*\text{-}\Gamma$ and has no finite-dimensional representations. We also give the definition of property MF- $c^*\text{-}\Gamma$. We then conclude that, for the unital MF C^* -algebra $\mathcal{A} = C^*(x_1, x_2, \dots, x_n)$ with no finite-dimensional representations, if \mathcal{A} has property MF- $c^*\text{-}\Gamma$, then $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$.

2. Definitions and Preliminaries

In this section, we are going to recall Voiculescu's definition of the topological free entropy dimension and topological orbit dimension of n -tuples of elements in a unital C^* -algebra.

2.1. A Covering of a set in a metric space. Suppose (X, d) is a metric space and K is a subset of X . A family of balls in X is called a covering of K if the union of these balls covers K and the centers of these balls lie in K .

2.2. Covering numbers in complex matrix algebra $(\mathcal{M}_k(\mathbb{C}))^n$. Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in \mathbb{C} , and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}Tr$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}(k)$ denote the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k(\mathbb{C}))^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k^{s,a}(\mathbb{C})$ be the subalgebra of $\mathcal{M}_k(\mathbb{C})$ consisting of all self-adjoint matrices of $\mathcal{M}_k(\mathbb{C})$. Let $(\mathcal{M}_k^{s,a}(\mathbb{C}))^n$ be the direct sum (or orthogonal sum) of n copies of $\mathcal{M}_k^{s,a}(\mathbb{C})$. Let $\|\cdot\|$ be an operator norm on $\mathcal{M}_k(\mathbb{C})^n$ defined by

$$\|(A_1, \dots, A_n)\| = \max\{\|A_1\|, \dots, \|A_n\|\}$$

for all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$. Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \dots, A_n)\|_2 = \sqrt{\tau_k(A_1^* A_1) + \dots + \tau_k(A_n^* A_n)}$$

for all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω - $\|\cdot\|$ -ball $Ball(B_1, \dots, B_n; \omega, \|\cdot\|)$ centered at (B_1, \dots, B_n) in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \dots, A_n) - (B_1, \dots, B_n)\| < \omega.$$

DEFINITION 1. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_\infty(\Sigma, \omega)$ to be the minimal number of ω - $\|\cdot\|$ -balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω - $\|\cdot\|_2$ -ball $Ball(B_1, \dots, B_n; \omega, \|\cdot\|_2)$ centered at (B_1, \dots, B_n) in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \dots, A_n) - (B_1, \dots, B_n)\|_2 < \omega.$$

DEFINITION 2. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_2(\Sigma, \omega)$ to be the minimal number of ω - $\|\cdot\|_2$ -balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

2.3. Unitary orbits of balls in $\mathcal{M}_k(\mathbb{C})^n$. For every $\omega > 0$, we define the ω -orbit- $\|\cdot\|$ -ball $\mathcal{U}(B_1, \dots, B_n; \omega)$ centered at (B_1, \dots, B_n) in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1, \dots, A_n) - (WB_1W^*, \dots, WB_nW^*)\| < \omega.$$

DEFINITION 3. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $o_\infty(\Sigma, \omega)$ to be the minimal number of ω -orbit- $\|\cdot\|$ -balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, we define the ω -orbit- $\|\cdot\|_2$ -ball $\mathcal{U}(B_1, \dots, B_n; \omega, \|\cdot\|_2)$ centered at (B_1, \dots, B_n) in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1, \dots, A_n) - (WB_1W^*, \dots, WB_nW^*)\|_2 < \omega.$$

DEFINITION 4. Suppose that Σ is a subset of $\mathcal{M}_k(\mathbb{C})^n$. We define $o_2(\Sigma, \omega)$ to be the minimal number of ω -orbit- $\|\cdot\|_2$ -balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

2.4. Noncommutative Polynomials. In this article, we always assume that

\mathcal{A} is a unital C^* -algebra. Let $x_1, \dots, x_n, y_1, \dots, y_m$ be self-adjoint elements in \mathcal{A} . Let $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$ be the set of all noncommutative polynomials in the indeterminates $X_1, \dots, X_n, Y_1, \dots, Y_m$. Let $\mathbb{C}_{\mathbb{Q}} = \mathbb{Q} + i\mathbb{Q}$ denote the complex-rational numbers, i.e., the number whose real and imaginary parts are rational. The set $\mathbb{C}_{\mathbb{Q}}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$ of noncommutative polynomials with complex-rational coefficients is countable. Throughout this paper we write

$$\mathbb{C}_{\mathbb{Q}}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle = \{P_r : r \in \mathbb{N}\} \text{ and } \mathbb{C}_{\mathbb{Q}}\langle X_1, \dots, X_n \rangle = \{Q_r : r \in \mathbb{N}\}$$

and

$$\mathbb{C}_{\mathbb{Q}}\langle X_1, X_2, \dots \rangle = \bigcup_{m=1}^{\infty} \mathbb{C}_{\mathbb{Q}}\langle X_1, \dots, X_m \rangle.$$

REMARK 1. We always assume that $1 \in \mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$.

2.5. Voiculescu's Norm-microstates Space. For all integers $r, k \geq 1$, real numbers $R, \varepsilon > 0$ and noncommutative polynomials P_1, \dots, P_r , we define

$$\Gamma_R^{(\text{top})}(x_1, \dots, x_n, y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r$$

to be the subset of $(\mathcal{M}_k^{s,a}(\mathbb{C}))^{n+m}$ consisting of all these

$$(A_1, \dots, A_n, B_1, \dots, B_m) \in (\mathcal{M}_k^{s,a}(\mathbb{C}))^{n+m}$$

satisfying

$$\max \{\|A_1\|, \dots, \|A_n\|, \|B_1\|, \dots, \|B_m\|\} \leq R$$

and

$$|\|P_j(A_1, \dots, A_n, B_1, \dots, B_m)\| - \|P_j(x_1, \dots, x_n, y_1, \dots, y_m)\|| \leq \varepsilon, \forall 1 \leq j \leq r.$$

REMARK 2. In the original definition of norm-microstates space in [24], the parameter R was not introduced. Note the following observation: Let

$$R > \max \{\|x_1\|, \|x_2\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}.$$

When r is large enough and ε is small enough,

$$\Gamma_R^{(\text{top})}(x_1, \dots, x_n, y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r) = \Gamma_{\infty}^{(\text{top})}(x_1, \dots, x_n, y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r)$$

for all $k \geq 1$. Our definition agrees with the one in [24] for large R, r and small ε .

Define the norm-microstates space of x_1, \dots, x_n in the presence of y_1, \dots, y_m , denoted by

$$\Gamma_R^{(\text{top})}(x_1, \dots, x_n : y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r)$$

as the projection of $\Gamma_R^{(\text{top})}(x_1, \dots, x_n, y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r)$ onto the space $(\mathcal{M}_k^{s,a}(\mathbb{C}))^n$ via the mapping

$$(A_1, \dots, A_n, B_1, \dots, B_m) \rightarrow (A_1, \dots, A_n).$$

2.6. Voiculescu's topological free entropy dimension. Define

$$\nu_\infty(\Gamma_R^{(\text{top})}(x_1, \dots, x_n : y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r), \omega)$$

to be the covering number of the set $\Gamma_R^{(\text{top})}(x_1, \dots, x_n : y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r)$ by ω - $\|\cdot\|$ -balls in the metric space $(M_k^{s.a}(\mathbb{C}))^n$ equipped with operator norm.

DEFINITION 5. *Define*

$$\begin{aligned} & \delta_{\text{top}}(x_1, \dots, x_n; \omega) \\ &= \sup_{R>0} \inf_{\varepsilon>0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(\nu_\infty(\Gamma_R^{(\text{top})}(x_1, \dots, x_n; k, \varepsilon, Q_1, \dots, Q_r), \omega))}{-k^2 \log \omega}. \end{aligned}$$

The topological free entropy dimension of x_1, \dots, x_n is defined by

$$\delta_{\text{top}}(x_1, \dots, x_n) = \limsup_{\omega \rightarrow 0^+} \delta_{\text{top}}(x_1, \dots, x_n; \omega).$$

Similarly, define

$$\begin{aligned} & \delta_{\text{top}}(x_1, \dots, x_n : y_1, \dots, y_m; \omega) \\ &= \sup_{R>0} \inf_{\varepsilon>0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(\nu_\infty(\Gamma_R^{(\text{top})}(x_1, \dots, x_n : y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r), \omega))}{-k^2 \log \omega}. \end{aligned}$$

The topological free entropy dimension of x_1, \dots, x_n in the presence of y_1, \dots, y_m is defined by

$$\delta_{\text{top}}(x_1, \dots, x_n : y_1, \dots, y_m) = \limsup_{\omega \rightarrow 0^+} \delta_{\text{top}}(x_1, \dots, x_n : y_1, \dots, y_m; \omega).$$

REMARK 3. *Let $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}$ be some positive number. By Remark 2, we know the supremum over $R > 0$ is unnecessary, i.e.,*

$$\begin{aligned} & \delta_{\text{top}}(x_1, \dots, x_n : y_1, \dots, y_m) \\ &= \limsup_{\omega \rightarrow 0^+} \inf_{\varepsilon>0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(\nu_\infty(\Gamma_R^{(\text{top})}(x_1, \dots, x_n : y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r), \omega))}{-k^2 \log \omega} \end{aligned}$$

2.7. $\delta_{\text{top}}^{1/2}$. In this subsection we recall the definition of $\delta_{\text{top}}^{1/2}$ and its properties.

DEFINITION 6. ([24]) *The norm-semi-microstates $\Gamma_{1/2}^{\text{top}}(x_1, \dots, x_n; k, \varepsilon, Q_1, \dots, Q_r)$ is the set of all $(a_1, \dots, a_n) \in \mathcal{M}_k^n(\mathbb{C})$ such that*

$$\|Q_j(a_1, \dots, a_n)\| \leq \|Q_j(x_1, \dots, x_n)\| + \varepsilon$$

for $1 \leq j \leq r$.

We define $\delta_{\text{top}}^{1/2}(x_1, \dots, x_n)$ to be

$$\limsup_{\omega \rightarrow 0^+} \inf_{r \in \mathbb{N}, \varepsilon>0} \limsup_{k \rightarrow \infty} \frac{\log\left(\nu_\infty\left(\Gamma_{1/2}^{\text{top}}((x_1, \dots, x_n; k, \varepsilon, Q_1, \dots, Q_r)), \omega\right)\right)}{-k^2 \log \omega}$$

THEOREM 1. ([18]) *$\delta_{\text{top}}^{1/2}(x_1, \dots, x_n) = \delta_{\text{top}}(x_1, \dots, x_n)$ whenever $\delta_{\text{top}}(x_1, \dots, x_n)$ is defined.*

2.8. Topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ and Modified free entropy dimension $\mathfrak{K}_2^{(2)}$. In this subsection, we are going to recall a C^* -algebra invariant "topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ " and its basic properties.

DEFINITION 7. ([11]) *Define*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n; \omega) = \sup_{R>0} \inf_{\varepsilon>0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n; k, \varepsilon, Q_1, \dots, Q_r), \omega))}{k^2}$$

and

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = \sup_{\omega>0} \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n; \omega) = \lim_{\omega \rightarrow 0^+} \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n; \omega).$$

Similarly, define

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \omega) = \sup_{R>0} \inf_{\varepsilon>0, r \in \mathbb{N}} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \dots, x_n : y_1, \dots, y_m; k, \varepsilon, P_1, \dots, P_r), \omega))}{k^2}$$

and

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m) = \sup_{\omega>0} \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n; \omega) = \lim_{\omega \rightarrow 0^+} \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \omega)$$

The topological orbit dimension $\mathfrak{K}_{top}^{(2)}$ is in fact a C^* -algebra invariant. In view of this result, we use $\mathfrak{K}_{top}^{(2)}(\mathcal{A})$ to denote $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n)$ for an arbitrary generating set $\{x_1, \dots, x_n\}$ for \mathcal{A} .

THEOREM 2. ([11]) *Suppose that \mathcal{A} is a unital C^* -algebra and $\{x_1, \dots, x_n\}, \{y_1, \dots, y_p\}$ are two families of self-adjoint generators of \mathcal{A} . Then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p)$$

After slightly modify the proof of Theorem 2, we can conclude that

THEOREM 3. *Suppose that \mathcal{A} is a unital C^* -algebra and $x_1, \dots, x_n, y_1, \dots, y_p, w_1, \dots, w_t$ are self-adjoint elements in \mathcal{A} . If $C^*(x_1, \dots, x_n) = C^*(y_1, \dots, y_p)$, then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : w_1, \dots, w_t) = \mathfrak{K}_{top}^{(2)}(y_1, \dots, y_p : w_1, \dots, w_t)$$

REMARK 4. *From the definition, it is clear that*

- (1) $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_p) \geq \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_p, y_{p+1});$
- (2) *If $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : x_1, \dots, x_{n+j}) = 0$ ($j \geq 0$), then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_{n-1} : x_1, \dots, x_{n+j}) = 0$$

Let \mathcal{M} be a von Neumann algebra with a tracial state τ , and let x_1, \dots, x_n be self-adjoint elements in \mathcal{M} . For any positive R and ε , and any $m, k \in \mathbb{N}$, let $\Gamma_R(x_1, \dots, x_n; m, k, \varepsilon; \tau)$ be the subset of $\mathcal{M}_k^{s.a.}(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k^{s.a.}(\mathbb{C})^n$ such that

$$\max_{1 \leq j \leq n} \|A_j\| \leq R \text{ and } |\tau_k(A_{i_1} \cdots A_{i_q}) - \tau(x_{i_1} \cdots x_{i_q})| < \varepsilon,$$

for all $1 \leq i_1, \dots, i_q \leq n$ and $1 \leq q \leq m$.

For any $\omega > 0$, let $o_2(\Gamma_R(x_1, \dots, x_n; m, k, \varepsilon; \tau), \omega)$ be the minimal number of ω -orbit- $\|\cdot\|_2$ -balls in $\mathcal{M}_k(\mathbb{C})^n$ that constitute a covering of $\Gamma_R(x_1, \dots, x_n; m, k, \varepsilon; \tau)$. Now we define, successively,

$$\begin{aligned} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \omega; \tau) &= \sup_{R>0} \inf_{m \in \mathbb{N}, \varepsilon > 0} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(x_1, \dots, x_n; m, k, \varepsilon; \tau)))}{k^2} \\ \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) &= \lim_{\omega \rightarrow 0^+} \sup \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \omega; \tau) \end{aligned}$$

where $\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau)$ is called the modified free orbit-dimension of x_1, \dots, x_n with respect to the tracial state τ [11].

REMARK 5. ([11]) Suppose x_1, \dots, x_n is a family of self-adjoint elements in a von Neumann algebra with a tracial state τ . Let $\mathfrak{K}_2(x_1, \dots, x_n; \tau)$ be the upper orbit dimension of x_1, \dots, x_n defined in Definition 1 of [16]. Then $\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) = 0$ if $\mathfrak{K}_2(x_1, \dots, x_n; \tau) = 0$.

2.9. MF-Traces and MF Nuclear Algebras. We note that the definition of $\delta_{top}(x_1, \dots, x_n)$ makes sense if and only if, for every $\varepsilon > 0$ and every $r, k_0 \in \mathbb{N}$, there is a $k \geq k_0$ such that

$$\Gamma^{(top)}(x_1, \dots, x_n; k, \varepsilon, Q_1, \dots, Q_r) \neq \emptyset.$$

In [15], it has shown that this is equivalent to $C^*(x_1, \dots, x_n)$ being an MF C*-algebra in the sense of Blackadar and Kirchberg [1]. A C*-algebra \mathcal{A} is an MF-algebra if \mathcal{A} can be embedded into $\Pi_{1 \leq k < \infty} \mathcal{M}_{m_k}(\mathbb{C}) / \sum_{1 \leq k < \infty} \mathcal{M}_{m_k}(\mathbb{C})$ for some increasing sequence $\{m_k\}$ of positive integers. In particular $C^*(x_1, \dots, x_n)$ is an MF-algebra if there is a sequence $\{m_k\}$ of positive integers and sequences $\{A_{1k}\}, \dots, \{A_{nk}\}$ with $A_{1k}, \dots, A_{nk} \in \mathcal{M}_{m_k}(\mathbb{C})$ such that

$$\lim_{k \rightarrow \infty} \|Q(A_{1k}, \dots, A_{nk})\| = \|Q(x_1, \dots, x_n)\|$$

for every *-polynomial $Q(t_1, \dots, t_n)$.

DEFINITION 8. ([18]) Suppose $\mathcal{A} = C^*(x_1, \dots, x_n)$ is an MF C*-algebra. A tracial state τ on \mathcal{A} is an MF-trace if there is sequence $\{m_k\}$ of positive integers and sequences $\{A_{1k}\}, \dots, \{A_{nk}\}$ with $A_{1k}, \dots, A_{nk} \in \mathcal{M}_{m_k}(\mathbb{C})$ such that, for every *-polynomial p ,

- (1) $\lim_{k \rightarrow \infty} \|Q(A_{1k}, \dots, A_{nk})\| = \|Q(x_1, \dots, x_n)\|$, and
- (2) $\lim_{k \rightarrow \infty} \tau_{m_k}(Q(A_{1k}, \dots, A_{nk})) = \tau(Q(x_1, \dots, x_n))$.

We let $\mathcal{TS}(\mathcal{A})$ denote the set of all tracial states on \mathcal{A} and $\mathcal{T}_{MF}(\mathcal{A})$ denote the set of all MF-traces on \mathcal{A} .

DEFINITION 9. ([18]) A C*-algebra $\mathcal{A} = C^*(x_1, \dots, x_n)$ is MF-nuclear if $\pi_\tau(\mathcal{A})''$ is hyperfinite for every $\tau \in \mathcal{T}_{MF}(\mathcal{A})$ where π_τ is the GNS representation of \mathcal{A} with respect to τ .

3. Relation between $\mathfrak{K}_{top}^{(2)}$ and $\mathfrak{K}_2^{(2)}$

DEFINITION 10. ([11]) Suppose that \mathcal{A} is a unital C*-algebra and $\mathcal{TS}(\mathcal{A})$ is the set of all tracial states of \mathcal{A} . Suppose that x_1, \dots, x_n is a family of self-adjoint elements in \mathcal{A} . Define

$$\mathfrak{K}_2^{(2)}(x_1, \dots, x_n) = \sup_{\tau \in \mathcal{TS}(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau)$$

THEOREM 4. ([11]) *Suppose that \mathcal{A} is a unital C^* -algebra and x_1, \dots, x_n is a family of self-adjoint generating elements in \mathcal{A} . Then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) \leq \mathfrak{K}_2^{(2)}(x_1, \dots, x_n)$$

We can generalize the preceding theorem as follows. The proof is similar to the proof in [11], but for completeness, we give its proof here.

THEOREM 5. *Let \mathcal{A} be a unital C^* -algebra and $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ be a family of self-adjoint generating elements in \mathcal{A} . Then*

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m) \leq \sup_{\tau \in \mathcal{T}_{MF}(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \tau)$$

where $\mathcal{T}_{MF}(\mathcal{A})$ is the set of all MF-tracial states on \mathcal{A} .

PROOF. If $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m) = 0$, the inequality holds automatically. Assume

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m) > \alpha > 0,$$

we need to show that $\mathfrak{K}_2^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \tau) > \alpha > 0$ for some MF trace τ . Let $\{P_r\}_{r=1}^\infty$ be a family of noncommutative polynomials in $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_m \rangle$ with rational coefficients. Let $R > \max\{\|x_1\|, \dots, \|x_n\|, \|y_1\|, \dots, \|y_m\|\}$. From the assumption that $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m) > \alpha$, it follows that there exist a positive number $\omega_0 > 0$ and a sequence of positive integers $\{k_q\}_{q=1}^\infty$ with $k_1 < k_2 < \dots$, so that for some $\alpha' > \alpha$

$$\lim_{q \rightarrow \infty} \frac{\log \left(\alpha_2 \left(\Gamma_R^{(top)} \left(x_1, \dots, x_n : y_1, \dots, y_m; k_q, \frac{1}{q}, P_1, \dots, P_q \right), \omega_0 \right) \right)}{k_q^2} > \alpha'.$$

Let $\mathcal{A}(n+m)$ be the universal unital C^* -algebra generated by self-adjoint elements

$$a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m}$$

of norm R , that is the unital full free product of $n+m$ copies of $C[-R, R]$. A microstate

$$\begin{aligned} \eta &= (A_1, \dots, A_n, B_1, \dots, B_m) \\ &\in \Gamma_R^{(top)} \left(x_1, \dots, x_n, y_1, \dots, y_m; k_q, \frac{1}{q}, P_1, \dots, P_q \right) = \Gamma(q) \end{aligned}$$

define a unital $*$ -homomorphism $\varphi_\eta : \mathcal{A}(n+m) \rightarrow \mathcal{M}_{k_q}(\mathbb{C})$ so that $\varphi_\eta(a_i) = A_i$ ($1 \leq i \leq n$) and $\varphi_\eta(a_i) = B_j$ ($n+1 \leq i \leq n+m$) as well as a tracial state $\tau_\eta \in \mathcal{TS}(\mathcal{A}(n+m))$ with $\tau_\eta = \frac{\text{Tr}_{k_q} \circ \varphi_\eta}{k_q}$. Similarly there is a $*$ -homomorphism $\varphi : \mathcal{A}(n+m) \rightarrow \mathcal{A}$ so that

$$\varphi(a_i) = x_i \quad (1 \leq i \leq n),$$

$$\text{and } \varphi(b_j) = y_j \quad (n+1 \leq i \leq n+m).$$

It is not hard to see that the weak topology on $\Omega = \mathcal{TS}(\mathcal{A}(n+m))$ is induced by the metric

$$d(\tau_1, \tau_2) = \sum_{s=1}^{\infty} \sum_{i_1, \dots, i_s \in \{1, \dots, n+m\}^s} (2R(n+m))^{-s} |(\tau_1 - \tau_2)(t_{i_1} \cdots t_{i_s})|$$

where $t_i \in \{a_1, \dots, a_{n+m}\}$. Therefore Ω is a compact metric space and

$$K_q = \{\tau_\eta \in \Omega \mid \eta \in \Gamma(q)\}$$

is a compact subset of Ω because $\eta \rightarrow \tau_\eta$ is continuous and $\Gamma(q)$ is compact. Let further $K \subseteq \Omega$ denote the compact subset $(\mathcal{TS}(\mathcal{A})) \circ \varphi$. Given $\varepsilon > 0$, from the fact that Ω is compact it follows that there is some $L(\varepsilon) > 0$ so that for each $q \geq 1$,

$$K_q = K_q^1 \cup \dots \cup K_q^{L(\varepsilon)}$$

where each compact set K_q^j has diameter $< \varepsilon$. Define

$$\Phi : \underbrace{\mathcal{M}_{k_q}(\mathbb{C}) \oplus \dots \oplus \mathcal{M}_{k_q}(\mathbb{C})}_{n+m} \longrightarrow \underbrace{\mathcal{M}_{k_q}(\mathbb{C}) \oplus \dots \oplus \mathcal{M}_{k_q}(\mathbb{C})}_n$$

by $\Phi(A_1, \dots, A_n, B_1, \dots, B_m) = (A_1, \dots, A_n)$. Let

$$\Gamma(q, j) = \{\eta \in \Gamma(q) \mid \tau_\eta \in K_q^j\}$$

We have $\Gamma(q) = \Gamma(q, 1) \cup \dots \cup \Gamma(q, L(\varepsilon))$. Let further $\Gamma'(q)$ denote some $\Gamma(q, j)$ such that

$$o_2(\Phi(\Gamma'(q)), \omega_0) \geq \frac{o_2(\Phi(\Gamma(q)), \omega_0)}{L(\varepsilon)}.$$

Thus we have $\lim_{q \rightarrow \infty} \frac{\log(o_2(\Phi(\Gamma'(q)), \omega_0))}{k_q^2} > \alpha'$. Given ε successively the values $1, 1/2, \dots, 1/s, \dots$, we can find a subsequence $\{q_s\}_{s=1}^\infty$ such that the chosen set $K_{q_s}^{j_s} \subseteq K_{q_s}$ has diameter $< \frac{1}{\varepsilon}$ and the corresponding set $\Gamma'(q_s)$ satisfying

$$\lim_{q \rightarrow \infty} \frac{\log(o_2(\Phi(\Gamma'(q_s)), \omega_0))}{k_{q_s}^2} > \alpha'$$

Without loss of generality, we can assume that τ is the weak limit of some sequence $(\tau_{\eta(q_s)})_{s=1}^\infty$. Then $\tau \in K$. In fact

$$|\tau(Q(a_1, \dots, a_n, b_1, \dots, b_m))| = \lim_{s \rightarrow \infty} |\tau_{\eta(q_s)}(Q(a_1, \dots, a_n, b_1, \dots, b_m))|,$$

therefore τ is an MF trace.

We can further assume that there is a subsequence $\{q_{s(t)}\}_{t=1}^\infty$ of $\{q_s\}_{s=1}^\infty$ so that the chosen set $K_{q_{s(t)}}^{j_s(t)} \subseteq K_{q_{s(t)}} \subseteq B(\tau, \frac{1}{t})$, the ball of radius $1/t$ and center τ . Therefore, for any $m \in \mathbb{N}$ and $\varepsilon > 0$, we have

$$\Gamma'(q_{s(t)}) \subseteq \Gamma_R(x_1, \dots, x_n, y_1, \dots, y_m; k_{q_{s(t)}}, m, \varepsilon; \tau)$$

when t is large enough. Thus $\Phi(\Gamma'(q_{s(t)})) \subseteq \Phi(\Gamma_R(x_1, \dots, x_n, y_1, \dots, y_m; k_{q_{s(t)}}, m, \varepsilon; \tau))$. Hence

$$\begin{aligned} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \tau) &\geq \mathfrak{K}_2^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \omega_0; \tau) \\ &\geq \lim_{t \rightarrow \infty} \frac{\log o_2(\Phi(\Gamma'(q_{s(t)})), \omega_0)}{k_{q_{s(t)}}^2} > \alpha', \end{aligned}$$

and hence

$$\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m) \leq \sup_{\tau \in \mathcal{T}_{MF}(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \dots, x_n : y_1, \dots, y_m; \tau).$$

□

Now we are ready to simplify the proof of the following theorem.

THEOREM 6. ([18]) *Suppose \mathcal{A} is an MF-nuclear C^* -algebra with a family of self-adjoint generators x_1, \dots, x_n . Then*

$$\delta_{top}(x_1, \dots, x_n) \leq 1.$$

PROOF. It is known that the GNS representation of an MF nuclear C^* -algebra with respect to an MF tracial state yields an injective von Neumann algebra. From [16] and Remark 5

$$\mathfrak{K}_2^{(2)}(x_1, \dots, x_n; \tau) = 0 \text{ for any } \tau \in \mathcal{TS}_{MF}(\mathcal{A})$$

where $\mathcal{TS}_{MF}(\mathcal{A})$ is the set of all MF tracial states of \mathcal{A} . So, from Theorem 5, we know that $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n) = 0$. Hence by Theorem 3.1.2 in [11],

$$\delta_{top}(x_1, \dots, x_n) \leq 1.$$

□

4. Topological free entropy dimension of inner quasidiagonal C^* -algebras

In this section, we will first recall the concept of inner quasidiagonal C^* -algebras which was first introduced by Blackadar and Kirchberg in [2]. After that, we are going to analyze the topological free entropy dimension of an MF-nuclear and inner quasidiagonal C^* -algebra.

DEFINITION 11. ([2]) *If \mathcal{B} is a C^* -algebra, then a projection $p \in \mathcal{B}$ is in the socle of \mathcal{B} if $p\mathcal{B}p$ is finite-dimensional. Denote the set of such projections by $\text{socle}(\mathcal{B})$.*

PROPOSITION 1. ([2]) *A C^* -algebra \mathcal{A} is inner quasidiagonal if and only if, for any $x_1, \dots, x_m \in \mathcal{A}$ and $\varepsilon > 0$, there is a projection $p \in \text{socle}(\mathcal{A}^{**})$ with $\|px_jp\| > \|x_j\| - \varepsilon$ and $\|px_j - x_jp\| < \varepsilon$ for all j .*

THEOREM 7. ([3]) *A separable C^* -algebra is inner QD if and only if it has a separating family of quasidiagonal irreducible representations.*

It is well-known that every residually finite-dimensional (RFD) C^* -algebra is inner quasidiagonal. In [18], it has shown that $\delta_{top}(x_1, \dots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}$ in which $\mathcal{A} = C^*(x_1, \dots, x_n)$ is MF-nuclear and residually finite-dimensional. Next theorem will generalize this result to inner quasidiagonal C^* -algebras with finite generators.

THEOREM 8. *Suppose $\mathcal{A} = C^*(x_1, \dots, x_n)$ is MF-nuclear and inner quasidiagonal, then $\delta_{top}(x_1, \dots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}$.*

PROOF. If $\dim \mathcal{A} < \infty$, then \mathcal{A} is RFD. So the conclusion is followed from Corollary 5.4 in [18]. Now suppose $\dim \mathcal{A} = \infty$. Since \mathcal{A} is MF-nuclear, we have $\delta_{top}(x_1, \dots, x_n) \leq 1$ by Theorem 6.

Let $\mathcal{F}_0 = \{x_1, \dots, x_n\}$. Suppose

$$\{1\} \cup \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$$

be the sequence of finite subsets of \mathcal{A} such that $\overline{\bigcup_i \mathcal{F}_i} = \mathcal{A}$. So by Theorem 7 and the property of quasidiagonal C^* -algebras, we can find a increasing sequence of projections $\{P_t\}_{t=0}^\infty \in \text{socle}(\mathcal{A}^{**})$ such that $\|P_t x - x P_t\| < \frac{\varepsilon}{2^t}$ and $\|P_t x P_t\| > \|x\| - \frac{\varepsilon}{2^t}$ as $x \in \mathcal{F}_t$. Note that

$$(P_i - P_{i-1})\mathcal{A}(P_i - P_{i-1}) \subseteq \mathcal{A}^{**}.$$

Since $P_t \mathcal{A}^{**} P_t = P_t \mathcal{A} P_t$, then

$$\begin{aligned} (P_i - P_{i-1}) \mathcal{A} (P_i - P_{i-1}) &= P_t ((P_i - P_{i-1}) \mathcal{A} (P_i - P_{i-1})) P_t \\ &\subseteq P_t \mathcal{A}^{**} P_t = P_t \mathcal{A} P_t \quad \text{as } t \geq i. \end{aligned}$$

Therefore

$$P_0 \mathcal{A} P_0 \oplus (P_1 - P_0) \mathcal{A} (P_1 - P_0) \oplus \cdots \oplus (P_t - P_{t-1}) \mathcal{A} (P_t - P_{t-1}) \subseteq P_t \mathcal{A} P_t$$

It follows that

$$\begin{aligned} \dim(P_t \mathcal{A} P_t) \\ \geq \dim(P_0 \mathcal{A} P_0 \oplus (P_1 - P_0) \mathcal{A} (P_1 - P_0) \oplus \cdots \oplus (P_t - P_{t-1}) \mathcal{A} (P_t - P_{t-1})) \end{aligned}$$

Then $\dim(P_t \mathcal{A} P_t) \rightarrow \infty$ as $t \rightarrow \infty$.

Note that $\|P_t x_j - x_j P_t\| \rightarrow 0$ and $\|P_t x_j P_t\| \rightarrow \|x_j\|$ as $t \rightarrow \infty$ for $1 \leq j \leq n$, so for any $0 < \varepsilon_0$ and $r_0 \in \mathbb{N}$, there are $\varepsilon_1, r_1, t_1 > 0$ such that for every $t > t_1, 0 < \varepsilon < \varepsilon_1$ and $r > r_1$

$$\Gamma_{1/2}^{\text{top}}(P_t x_1 P_t, \dots, P_t x_n P_t; k, \varepsilon, Q_1, \dots, Q_r) \subseteq \Gamma_{1/2}^{\text{top}}(x_1, \dots, x_n; k, \varepsilon_0, Q_1, \dots, Q_{r_0})$$

for every $k \in \mathbb{N}$. Therefore for any $\omega > 0$

$$\begin{aligned} (4.1) \quad & \sup_{t > t_1} \inf_{\varepsilon < \varepsilon_1, r > r_1} \limsup_{k \rightarrow \infty} \frac{\log \left(v_{\|\cdot\|} \left(\Gamma_{1/2}^{\text{top}}(P_t x_1 P_t, \dots, P_t x_n P_t; k, \varepsilon, Q_1, \dots, Q_r), \omega \right) \right)}{-k^2 \log \omega} \\ & \leq \limsup_{k \rightarrow \infty} \frac{\log \left(v_{\|\cdot\|} \left(\Gamma_{1/2}^{\text{top}}(x_1, \dots, x_n; k, \varepsilon_0, Q_1, \dots, Q_{r_0}), \omega \right) \right)}{-k^2 \log \omega}. \end{aligned}$$

It is obvious that $C^*(P_t x_1 P_t, \dots, P_t x_n P_t) \subseteq P_t \mathcal{A} P_t$ by the fact that $P_t \in \text{socle}(\mathcal{A}^{**})$. Let $\dim(P_{t_0} \mathcal{A} P_{t_0}) = M_{t_0} < \infty$. Then we can find $\{a_1, \dots, a_{M_{t_0}}\} \subseteq \mathcal{A}$ such that

$$\{P_{t_0} a_1 P_{t_0}, \dots, P_{t_0} a_{M_{t_0}} P_{t_0}\}$$

is a linearly independent family where $P_{t_0} a_i P_{t_0}$ is an element with norm 1 for every $i = 1, \dots, M_{t_0}$. For any $\varepsilon > 0$, there are polynomials

$$H_1(X_1, \dots, X_n), \dots, H_{M_{t_0}}(X_1, \dots, X_n)$$

in X_1, \dots, X_n such that

$$\|a_i - H_i(x_1, \dots, x_n)\| < \varepsilon \text{ for every } 1 \leq i \leq M_{t_0}$$

It follows that

$$\|P_t a_i P_t - P_t H_i(x_1, \dots, x_n) P_t\| < \varepsilon.$$

Since $\|P_t x_j - x_j P_t\| \rightarrow 0$ as $t \rightarrow \infty$ for every $1 \leq j \leq n$, then there is an integer L such that

$$\|P_t a_i P_t - H_i(P_t x_1 P_t, \dots, P_t x_n P_t)\| < L \cdot \varepsilon \text{ for every } 1 \leq i \leq M_{t_0}$$

as $t > t_0$ big enough. Hence

$$\begin{aligned} & \|P_{t_0} a_i P_{t_0} - P_{t_0} H_i(P_t x_1 P_t, \dots, P_t x_n P_t) P_{t_0}\| \\ & = \|P_{t_0} P_t a_i P_t P_{t_0} - P_{t_0} H_i(P_t x_1 P_t, \dots, P_t x_n P_t) P_{t_0}\| < L \cdot \varepsilon \end{aligned}$$

Note that ε is arbitrary, $\{P_{t_0} a_1 P_{t_0}, \dots, P_{t_0} a_{M_{t_0}} P_{t_0}\}$ is a linear basis of $P_{t_0} \mathcal{A} P_{t_0}$ and

$$P_{t_0} C^*(P_t x_1 P_t, \dots, P_t x_n P_t) P_{t_0} \subseteq P_{t_0} \mathcal{A} P_{t_0},$$

then

$$\{P_{t_0}H_1(P_tx_1P_t, \dots, P_tx_nP_t)P_{t_0}, \dots, P_{t_0}H_{M_{t_0}}(P_tx_1P_t, \dots, P_tx_nP_t)P_{t_0}\}$$

is a linearly independent family in $P_{t_0}C^*(P_tx_1P_t, \dots, P_tx_nP_t)P_{t_0}$ as t big enough. Therefore

$$\dim P_{t_0}C^*(P_tx_1P_t, \dots, P_tx_nP_t)P_{t_0} = M_{t_0}.$$

It follows that, for such t ,

$$\begin{aligned} \dim C^*(P_tx_1P_t, \dots, P_tx_nP_t) &\geq \dim P_{t_0}C^*(P_tx_1P_t, \dots, P_tx_nP_t)P_{t_0} \\ &= M_{t_0} = \dim(P_{t_0}\mathcal{A}P_{t_0}) \end{aligned}$$

Let $N_t = \dim C^*(P_tx_1P_t, \dots, P_tx_nP_t)$. Then

$$N_t = \dim C^*(P_tx_1P_t, \dots, P_tx_nP_t) \rightarrow \infty$$

as $t \rightarrow \infty$ since $M_t = \dim(P_t\mathcal{A}P_t) \rightarrow \infty$ as $t \rightarrow \infty$. Note that

$$\delta_{top}(P_tx_1P_t, \dots, P_tx_nP_t) = 1 - \frac{1}{N_t}.$$

Then, for every t , there exists ω_t such that

$$\begin{aligned} &1 - \frac{1}{N_t - 1} \\ &\leq \inf_{\varepsilon, r} \limsup_{k \rightarrow \infty} \frac{v_{\|\cdot\|} \left(\Gamma_{1/2}^{\text{top}}(P_tx_1P_t, \dots, P_tx_nP_t; k, \varepsilon, p_1, \dots, p_r), \omega_t \right)}{-k^2 \log \omega_t}. \end{aligned}$$

Then, for every t , we have

$$\begin{aligned} &1 - \frac{1}{N_t - 1} \\ &\leq \inf_{\varepsilon_0, r_0} \limsup_{k \rightarrow \infty} \frac{v_{\|\cdot\|} \left(\Gamma_{1/2}^{\text{top}}(x_1, \dots, x_n; k, \varepsilon_0, p_1, \dots, p_{r_0}), \omega_t \right)}{-k^2 \log \omega_t} \end{aligned}$$

for ε_0, r_0 by (4.1). Therefore

$$\begin{aligned} &\lim_{t \rightarrow \infty} \left(1 - \frac{1}{N_t - 1} \right) \\ &\leq \limsup_{\omega_t} \inf_{\varepsilon_0, r_0} \limsup_{k \rightarrow \infty} \frac{v_{\|\cdot\|} \left(\Gamma_{1/2}^{\text{top}}(x_1, \dots, x_n; k, \varepsilon_0, p_1, \dots, p_{r_0}), \omega_t \right)}{-k^2 \log \omega_t} \\ &\leq \limsup_{\omega \rightarrow 0} \inf_{\varepsilon_0, r_0} \limsup_{k \rightarrow \infty} \frac{v_{\|\cdot\|} \left(\Gamma_{1/2}^{\text{top}}(x_1, \dots, x_n; k, \varepsilon_0, p_1, \dots, p_{r_0}), \omega \right)}{-k^2 \log \omega} \\ &= \delta_{top}(x_1, \dots, x_n) \end{aligned}$$

Hence $\delta_{top}(x_1, \dots, x_n) = 1 = 1 - \frac{1}{\dim \mathcal{A}}$ as $\dim \mathcal{A} = \infty$. This completes the proof. \square

REMARK 6. From [2], we know that a unital C^* -algebra $\mathcal{A} = C^*(x_1, \dots, x_n)$ is strong NF if and only if it is nuclear and inner QD. Therefore we have the following corollary:

COROLLARY 1. Let $\mathcal{A} = C^*(x_1, \dots, x_n)$ be a strong NF C^* -algebra. Then $\delta_{top}(x_1, \dots, x_n) = 1 - \frac{1}{\dim \mathcal{A}}$.

5. Definition and properties of $\mathfrak{K}_{top}^{(3)}$

Suppose \mathcal{A} is a unital C*-algebra.

Let $\infty \cdot 0 = 0$. For any subset $\mathcal{G} \subseteq \mathcal{A}$, define

$$\mathfrak{K}_{top}^{(3)}(x_1, \dots, x_n; \mathcal{G}) = \inf \left\{ \infty \cdot \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_t) : \{y_1, \dots, y_t\} \text{ is a finite subset of } \mathcal{G} \right\},$$

$$\mathfrak{K}_{top}^{(3)}(\mathcal{G}) = \sup_{\substack{E \subseteq \mathcal{G} \\ E \text{ is finite}}} \inf_{\substack{F \subseteq \mathcal{G} \\ F \text{ is finite}}} \mathfrak{K}_{top}^{(3)}(E : F)$$

REMARK 7. When \mathcal{G} is finite, it is not difficult to see that

$$\mathfrak{K}_{top}^{(3)}(x_1, \dots, x_n; \mathcal{G}) = \infty \cdot \mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n; \mathcal{G})$$

and

$$\mathfrak{K}_{top}^{(3)}(\mathcal{G}) = \infty \cdot \mathfrak{K}_{top}^{(2)}(\mathcal{G})$$

The proof of the following theorem is similar to the Theorem 3.3 in [12], so we omit it.

THEOREM 9. If \mathcal{A} is an MF-algebra, then the following are equivalent:

- (1) $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$;
- (2) if $x_1, \dots, x_n \in \mathcal{A}$, then there exist $y_1, \dots, y_t \in \mathcal{A}$ such that $\mathfrak{K}_{top}^{(2)}(x_1, \dots, x_n : y_1, \dots, y_t) = 0$;
- (3) for any generating set \mathcal{G} of \mathcal{A} , $\mathfrak{K}_{top}^{(3)}(\mathcal{G}) = 0$;
- (4) there exists a generating set \mathcal{G} of \mathcal{A} such that $\mathfrak{K}_{top}^{(3)}(\mathcal{G}) = 0$;
- (5) if \mathcal{G} is a generating set of \mathcal{A} , and A_0 is a finite subset of \mathcal{G} , then, for any finite subset A with $A_0 \subseteq A \subseteq \mathcal{G}$, there exists a finite subset B of \mathcal{G} so that $\mathfrak{K}_{top}^{(3)}(A : B) = 0$.
- (6) there is an increasing directed family $\{\mathcal{A}_i : i \in \Lambda\}$ of C*-subalgebras of \mathcal{A} such that
 - (a) each \mathcal{A}_i is countably generated;
 - (b) $\mathfrak{K}_{top}^{(3)}(\mathcal{A}_i) = 0$;
 - (c) $\mathcal{A} = \cup_{i \in \Lambda} \mathcal{A}_i$
- (7) If A is a countable subset of \mathcal{M} , then there exists a countably generated subalgebra \mathcal{B} of \mathcal{M} such that $A \subseteq \mathcal{B}$ and $\mathfrak{K}_{top}^{(3)}(\mathcal{B}) = 0$

REMARK 8. If \mathcal{A} is finite generated, then $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$ if

$$\mathfrak{K}_{top}^{(2)}(\mathcal{A}) = 0.$$

COROLLARY 2. Suppose \mathcal{A} is a C*-algebra, \mathcal{G} is a generating set of \mathcal{A} . Then $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = \mathfrak{K}_{top}^{(3)}(\mathcal{G})$.

COROLLARY 3. Suppose $\{\mathcal{A}_l\}_{l \in \Lambda}$ is an increasingly directed family of C*-algebras. Then

$$\mathfrak{K}_{top}^{(3)}(\cup_l \mathcal{A}_l) \leq \liminf_l \mathfrak{K}_{top}^{(3)}(\mathcal{A}_l).$$

PROOF. If $\liminf_l \mathfrak{K}_{top}^{(3)}(\mathcal{A}_l) = \infty$, the inequality holds clearly. Suppose that $\liminf_l \mathfrak{K}_{top}^{(3)}(\mathcal{A}_l) = 0$. Let $x_1, \dots, x_n \in \cup \mathcal{A}_l$. Then we can find a $\lambda \in \Lambda$ such that $x_1, \dots, x_n \in \mathcal{A}_\lambda$ and $\mathfrak{K}_{top}^{(3)}(\mathcal{A}_l) = 0$. Therefore, we can find $y_1, \dots, y_p \in \mathcal{A}_\lambda$ with

$$\mathfrak{K}_{top}^{(3)}(x_1, \dots, x_n : y_1, \dots, y_p) = 0.$$

It follows that $\mathfrak{K}_{top}^{(3)}(\cup \mathcal{A}_l) = 0$ by Theorem 9 (6). \square

We modify the \mathfrak{K}_3 in [12] by using the modified free orbit dimension $\mathfrak{K}_2^{(2)}$.

DEFINITION 12. Let \mathcal{M} be a von Neumann algebra with a tracial state τ , and let x_1, \dots, x_n be self-adjoint elements in \mathcal{M} . Define

$$\begin{aligned} & \mathfrak{K}_3^{(3)}(x_1, \dots, x_n; \mathcal{G}) \\ &= \inf \left\{ \infty \cdot \mathfrak{K}_2^{(2)}(x_1, \dots, x_n : y_1, \dots, y_t) : \{y_1, \dots, y_t\} \text{ is a finite subset of } \mathcal{G} \right\} \end{aligned}$$

and

$$\mathfrak{K}_3^{(3)}(\mathcal{G}) = \sup_{\substack{E \subseteq \mathcal{G} \\ E \text{ is finite}}} \inf_{\substack{F \subseteq \mathcal{G} \\ F \text{ is finite}}} \mathfrak{K}_3^{(3)}(E : F)$$

REMARK 9. Note that $\mathfrak{K}_3(\mathcal{G}) = 0$, then $\mathfrak{K}_3^{(3)}(\mathcal{G}) = 0$ by Remark 5.

THEOREM 10. Let \mathcal{A} be an MF-algebra and \mathcal{G} be a generating set of \mathcal{A} . Then

$$\mathfrak{K}_{top}^{(3)}(\mathcal{G}) \leq \sup_{\tau \in \mathcal{TS}(\mathcal{A})} \mathfrak{K}_3^{(3)}(\mathcal{G} : \tau)$$

PROOF. If $\mathfrak{K}_{top}^{(3)}(\mathcal{G}) = 0$, the inequality holds clearly. Now suppose $\mathfrak{K}_{top}^{(3)}(\mathcal{G}) = \infty$, then there is a subset $E \subseteq \mathcal{G}$ such that for every finite subset $F \subseteq \mathcal{G}$, $\mathfrak{K}_{top}^{(3)}(E : F) = \infty$. Therefore $0 < \mathfrak{K}_{top}^{(2)}(E : F) \leq \infty$ for every finite subset $F \subseteq \mathcal{G}$. Let F be an arbitrary finite subset of \mathcal{G} . Then we can find a sequence $\{F_i\}_{i=1}^\infty$ of finite subset of \mathcal{G} with

$$F = F_0 \subseteq F_1 \subseteq \dots$$

and $\cup_i F_i = \mathcal{G}$. Hence

$$C^*(E \cup F_0) \subseteq C^*(E \cup F_1) \subseteq \dots$$

and

$$\overline{\cup_i C^*(E \cup F_i)}^{\|\cdot\|} = \mathcal{A}.$$

Now for each i , $\mathfrak{K}_{top}^{(2)}(E : F_i) > 0$. So by Theorem 5, we may find a tracial state τ_i on $C^*(E \cup F_i)$ such that

$$\mathfrak{K}_2^{(2)}(E : F_i; \tau_i) \geq \mathfrak{K}_{top}^{(2)}(E : F_i) - \varepsilon_i > 0$$

for some small $\varepsilon_i > 0$ for each $i \in \mathbb{N}$. If we regard $C^*(E \cup F_0)$ as a subalgebra of $C^*(E \cup F_i)$ for every $i \geq 1$, then

$$(5.1) \quad \mathfrak{K}_2^{(2)}(E : F_0; \tau_i) \geq \mathfrak{K}_2^{(2)}(E : F_i; \tau_i) > 0 \text{ for every } i \in \mathbb{N}.$$

Let

$$\pi : \overline{\cup_i C^*(E \cup F_i)}^{\|\cdot\|} \longrightarrow \Pi_{i \in \mathbb{N}} C^*(E \cup F_i) / \Sigma_{i \in \mathbb{N}} C^*(E \cup F_i)$$

be the embedding defined by

$$\pi(A) = \underbrace{(0, \dots, 0)}_{\text{the first } i \text{ positions}}, A, A, \dots) \text{ for every } A \in C^*(E \cup F_i).$$

and $\tilde{\tau}$ be the tracial state define by $\tilde{\tau}([A_i]_i) = \lim_{i \rightarrow \alpha} \tau_i(A_i)$ on

$$\Pi_{i \in \mathbb{N}} C^*(E \cup F_i) / \Sigma_{i \in \mathbb{N}} C^*(E \cup F_i)$$

where α is a free ultrafilter. Define $\tau = \tilde{\tau} \circ \pi$, then τ is a tracial state on \mathcal{A} . Note for any finite subset G of \mathcal{G} , we can always find a suitable index i such that

$$G \subseteq F_i \subseteq F_{i+1} \subseteq \dots.$$

Therefore $\tilde{\tau}$ is irrelevant to the selection of finite subset $F = F_0$, so is τ . Let $\{\varepsilon_t\}$ be a decreasing sequence of positive number with $\lim_{t \rightarrow \infty} \varepsilon_t = 0$ and $\{m_t\}_{t=1}^\infty$ be an increasing sequence of integers with $\lim_{t \rightarrow \infty} m_t = \infty$. Then, for every $R > 0$, we can find a subsequence $\{i_t\}_{t=1}^\infty$ of integers such that when ε is small enough and m is big enough, we always have

$$\Gamma_R(E : F_0; k, \varepsilon, m; \tau_{i_t}) \subseteq \Gamma_R(E : F_0; k, \varepsilon_t, m_t; \tau).$$

It implies that, for any $\omega > 0$,

$$\begin{aligned} & \inf_{\varepsilon, m} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(E : F_0; k, \varepsilon, m; \tau_{i_t}), \omega))}{k^2} \\ & \leq \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(E : F_0; k, \varepsilon_t, m_t; \tau), \omega))}{k^2}, \end{aligned}$$

it follows that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \inf_{\varepsilon, m} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(E : F_0; k, \varepsilon, m; \tau_{i_t}), \omega))}{k^2} \\ & \leq \limsup_{t \rightarrow \infty} \sup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(E : F_0; k, \varepsilon_t, m_t; \tau), \omega))}{k^2}, \end{aligned}$$

Therefore we can find an index t_0 such that

$$\begin{aligned} 0 < \mathfrak{K}_2^{(2)}(E : F_0 : \tau_{i_{t_0}}) &= \sup_{0 < \omega < 1} \sup_{R > 0} \inf_{\varepsilon, m} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(E : F_0; k, \varepsilon, m; \tau_{i_{t_0}}), \omega))}{k^2} \\ &\leq \sup_{0 < \omega < 1} \sup_{R > 0} \limsup_{t \rightarrow \infty} \inf_{\varepsilon, m} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(E : F_0; k, \varepsilon, m; \tau_{i_t}), \omega))}{k^2} \\ &\leq \sup_{0 < \omega < 1} \sup_{R > 0} \limsup_{t \rightarrow \infty} \sup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(E : F_0; k, \varepsilon_t, m_t; \tau), \omega))}{k^2} \\ &= \sup_{0 < \omega < 1} \sup_{R > 0} \inf_{\varepsilon_t, m_t} \limsup_{k \rightarrow \infty} \frac{\log(o_2(\Gamma_R(E : F_0; k, \varepsilon_t, m_t; \tau), \omega))}{k^2} \\ &= \mathfrak{K}_2^{(2)}(E : F_0 : \tau) \end{aligned}$$

Note that $F_0 = F$ is an arbitrary subset of \mathcal{G} . It implies that

$$\begin{aligned} \mathfrak{K}_3^{(3)}(\mathcal{G} : \tau) &= \sup_{\substack{E \subseteq \mathcal{G} \\ E \text{ is finite}}} \inf_{\substack{F \subseteq \mathcal{G} \\ F \text{ is finite}}} \mathfrak{K}_3^{(3)}(E : F; \tau) \\ &= \sup_{\substack{E \subseteq \mathcal{G} \\ E \text{ is finite}}} \inf_{\substack{F \subseteq \mathcal{G} \\ F \text{ is finite}}} \left\{ \infty \cdot \mathfrak{K}_2^{(2)}(E : F : \tau) \right\} = \infty. \end{aligned}$$

It follows that

$$\mathfrak{K}_{top}^{(3)}(\mathcal{G}) \leq \sup_{\tau \in \mathcal{TS}(\mathcal{A})} \mathfrak{K}_3^{(3)}(\mathcal{G} : \tau).$$

□

LEMMA 1. ([15]) Suppose x is a self-adjoint element in a unital C^* -algebra \mathcal{A} . Let $\sigma(x)$ be the spectrum of x in \mathcal{A} and $R > \|x\|$. For any $\varepsilon > 0$, we have the following results.

- (1) There are some integer $n \geq 1$ and distinct real numbers $\lambda_1, \dots, \lambda_n$ in $\sigma(x)$ satisfying
 - (a) $|\lambda_i - \lambda_j| \geq \omega$ for all $1 \leq i \neq j \leq n$; and
 - (b) for any $\lambda \in \sigma(x)$, there is some λ_j with $1 \leq j \leq n$ such that $|\lambda - \lambda_j| \leq \omega$.
- (2) There is some $r_0 \in \mathbb{N}$ such that the following holds: when $r \geq r_0$ for any $k \in \mathbb{N}$ and any $A \in \Gamma_R^{(top)}(x; k, r, \frac{1}{r})$, there are positive integers $1 \leq k_1, \dots, k_n \leq k$ with $k_1 + \dots + k_n = k$ and a unitary matrix $U \in \mathcal{U}(k)$ satisfying

$$\left\| U^* A U - \begin{pmatrix} \lambda_1 I_{k_1} & 0 & 0 & 0 \\ 0 & \lambda_2 I_{k_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n I_{k_n} \end{pmatrix} \right\| \leq 2\omega$$

where I_{k_j} is the $k_j \times k_j$ identity matrix in $\mathcal{M}_{k_j}(\mathbb{C})$ for $1 \leq j \leq n$.

LEMMA 2. ([8]) If \mathbb{B} is the unit ball in \mathbb{R}^m , then, with respect to the Euclidean metric

$$\left(\frac{1}{\omega} \right)^m \leq v(\mathbb{B}, \omega) \leq \left(\frac{3}{\omega} \right)^m$$

LEMMA 3. Let $\{\lambda_1, \dots, \lambda_k\} \subseteq [0, 1]$ with $\lambda_1 = \frac{1}{2k}$ and $|\lambda_i - \lambda_{i+1}| = \frac{1}{k}$ for $i = 1, \dots, k-1$. Assume D_1 and D_2 are diagonal matrices in $\mathcal{M}_k(\mathbb{C})$ with diagonal entries are all from $\{\lambda_1, \dots, \lambda_k\}$ without repetition. For every $\delta > 0$, let

$$\Omega(D_1, D_2; \delta) = \{U \in \mathcal{U}(k) \mid \|UD_1 - D_2U\|_2 \leq \delta\}.$$

Then, for every $0 < \delta < r$, there exists a set $\{\text{Ball}(U_\lambda; \frac{4\delta}{r})\}_{\lambda \in \Lambda}$ of $\frac{4\delta}{r}$ -balls in $\mathcal{U}(k)$ that cover $\Omega(D_1, D_2; \delta)$ with the cardinality of Λ satisfying $|\Lambda| \leq \left(\frac{3r}{2\delta}\right)^{4rk^2}$.

PROOF. Let $D = \text{diag}(\lambda_1, \dots, \lambda_k)$. Then there exist $W_1, W_2 \in \mathcal{U}(k)$ such that $D_1 = W_1 D W_1^*$ and $D_2 = W_2 D W_2^*$. Let

$$\tilde{\Omega}(\delta) = \{U \in \mathcal{U}(k) \mid \|UD - DU\|_2 \leq \delta\}.$$

Clearly

$$\Omega(D_1, D_2; \delta) = \{W_2^* U W_1 \mid U \in \tilde{\Omega}(\delta)\},$$

hence $\tilde{\Omega}(\delta)$ and $\Omega(D_1, D_2; \delta)$ have the same covering numbers.

Let $\{e_{st}\}_{s,t=1}^k$ be the canonical system of matrix units of $\mathcal{M}_k(\mathbb{C})$. Let

$$\mathcal{S}_1 = \text{span}\{e_{st} : |\lambda_s - \lambda_t| < r\}, \mathcal{S}_2 = \mathcal{M}_k(\mathbb{C}) \ominus \mathcal{S}_1.$$

For every $U = \sum_{s,t=1}^k x_{st} e_{st}$ in $\tilde{\Omega}(\delta)$, with $x_{st} \in \mathbb{C}$, let $T_1 = \sum_{e_{st} \in \mathcal{S}_1} x_{st} e_{st}$ and $T_2 = \sum_{e_{st} \in \mathcal{S}_2} x_{st} e_{st} \in \mathcal{S}_2$. But

$$\begin{aligned} \delta^2 &\geq \|UD - DU\|_2^2 = \sum_{s,t=1}^k |(\lambda_s - \lambda_t) x_{st}|^2 \geq \sum_{e_{st} \in \mathcal{S}_2} |(\lambda_s - \lambda_t) x_{st}|^2 \\ &\geq r^2 \sum_{e_{st} \in \mathcal{S}_2} |x_{st}|^2 = r^2 \|T_2\|_2^2. \end{aligned}$$

Hence $\|T_2\|_2 \leq \frac{\delta}{r}$.

Suppose $\lambda \in [0, 1]$. The number of the points in $\{\lambda_1, \dots, \lambda_k\}$ which lie inside the interval $(\lambda - r, \lambda + r)$ is at most the cardinality $m \neq 0$ of $(\lambda - r, \lambda + r) \cap \frac{1}{k}\mathbb{Z}$. Then $\frac{1}{k} \leq 2r$, this interval contains at least one point in $\frac{1}{k}\mathbb{Z}$, and an interval of length $4r$ should contain at least $m + 1$ such consecutive points. Hence $4r$ should be at least the length of the interval defined by $m + 1$ (i.e., $m\frac{1}{k}$) consecutive points in $\frac{1}{k}\mathbb{Z}$, i.e., $m\frac{1}{k} \leq 4r$. It follows that $\dim_{\mathbb{R}} \mathcal{S}_1 \leq 4rk^2$. Note that $\|T_1\|_2 \leq \|U\|_2 = 1$. Then $\tilde{\Omega}(\delta)$ can be covered by a set $\{Ball(A^\lambda; \frac{2\delta}{r})\}_{\lambda \in \Lambda}$ of $\frac{2\delta}{r}$ -balls in $\mathcal{M}_k(\mathbb{C})$ with $|\Lambda| \leq (\frac{3r}{2\delta})^{4rk^2}$ by Lemma 2. Because $\tilde{\Omega}(\delta) \subseteq \mathcal{U}(k)$, after replacing A^λ by a unitary U^λ in $Ball(U^\lambda; \frac{4\delta}{r})$, we obtain that the set $\{Ball(U^\lambda; \frac{4\delta}{r})\}_{\lambda \in \Lambda}$ of $\frac{4\delta}{r}$ -balls in $\mathcal{U}(k)$ that cover $\tilde{\Omega}(\delta)$ with the cardinality of Λ satisfying $|\Lambda| \leq (\frac{3r}{2\delta})^{4rk^2}$. The same result holds for $\Omega(D_1, D_2; \delta)$. \square

The following lemma can be found in [8]:

LEMMA 4. *Suppose A is a normal element in a von Neumann algebra \mathcal{M} with tracial state τ such that A has no eigenvalues. Then there is a positive element Y with the uniform distribution on $[0, 1]$ such that $W^*(A) = W^*(Y)$.*

REMARK 10. *It is well-known that every selfadjoint element of a finite von Neumann algebra \mathcal{M} has an eigenvalue if and only if \mathcal{M} has a finite-dimensional invariant subspace.*

LEMMA 5. *Let $x_1, \dots, x_n, y_1, \dots, y_p, v_1, \dots, v_s, w_1, \dots, w_t$ be elements in an MF C*-algebra \mathcal{A} . If $C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p)$ has no finite-dimensional representations, then*

$$\begin{aligned} &\mathfrak{K}_{top}^{(3)}(x_1, \dots, x_n, y_1, \dots, y_p : v_1, \dots, v_s, w_1, \dots, w_t) \\ &\leq \mathfrak{K}_{top}^{(3)}(x_1, \dots, x_n : v_1, \dots, v_s) + \mathfrak{K}_{top}^{(3)}(y_1, \dots, y_p : w_1, \dots, w_t) \end{aligned}$$

PROOF. Without loss of generality, we may assume that $\|x_i\| \leq 1$ and $\|y_j\| \leq 1$ for each $1 \leq i \leq n$ and $1 \leq j \leq p$. If one of $\mathfrak{K}_{top}^{(3)}(x_1, \dots, x_n : v_1, \dots, v_s)$ and $\mathfrak{K}_{top}^{(3)}(y_1, \dots, y_p : w_1, \dots, w_t)$ is infinity, then we are done. Now suppose

$$\mathfrak{K}_{top}^{(3)}(x_1, \dots, x_n : v_1, \dots, v_s) = \mathfrak{K}_{top}^{(3)}(y_1, \dots, y_p : w_1, \dots, w_t) = 0.$$

Let τ be a trace on $C^*(x_1, \dots, x_n, y_1, \dots, y_p, v_1, \dots, v_s, w_1, \dots, w_t)$. Let (π, M, e) denote the GNS construction for τ , i.e., $\pi : \mathcal{A} \rightarrow \mathcal{B}(M)$ is a unital *-homomorphism with unit cyclic vector e , such that, for every $a \in \mathcal{A}$, we have $\tau(a) = (\pi(a)e, e)$. Note the state ρ defined by $\rho(a) = (\pi(a)e, e)$ is faithful on

$$\pi(C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p))'',$$

so $\pi(C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p))''$ is finite. Since $C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p)$ has no finite-dimensional representation,

$$\pi(C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p))''$$

has no non-zero finite-dimensional invariant subspace. Hence there is an

$$a = a^* \in \pi(C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p))''$$

such that a has no eigenvalues by Remark 10. Let $\{a_k\}$ be a sequence of self-adjoint elements in $\pi(C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p))$ with $a_k \rightarrow a$ in the $*$ -strong operator topology. Then Voiculescu ([26]) proved that

$$1 = \delta_0(a, \rho) \leq \liminf_{k \rightarrow \infty} \delta_0(a_k, \rho) = \liminf_{k \rightarrow \infty} \left(1 - \sum_{t \text{ is an eigenvalue of } a} \rho(P_t)^2 \right)$$

where δ_0 is the free entropy dimension for von Neumann algebras. So we can find a self-adjoint element $a_k \in \pi(C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p))$ satisfying that a_k has no eigenvalues. Therefore there is an interval $[\alpha, \beta]$ such that $[\alpha, \beta] \subseteq \sigma(a_k)$. Let b be a self-adjoint element in $C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p)$ satisfying $\pi(b) = a_k$. Then $[\alpha, \beta] \subseteq \sigma(b)$. Define a continuous function

$$f(t) = \begin{cases} 0 & , -\|b\| \leq x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \alpha \leq x \leq \beta \\ 1 & \beta < x \leq \|b\| \end{cases}$$

over $[-\|b\|, \|b\|]$. Then $\sigma(f(b)) = [0, 1]$ and

$$f(b) \in C^*(x_1, \dots, x_n) \cap C^*(y_1, \dots, y_p).$$

Denote $d = f(b)$. If

$$(A_1, \dots, A_n, B_1, \dots, B_p, D)$$

$$\in \Gamma^{(top)}(x_1, \dots, x_n, y_1, \dots, y_p, d : v_1, \dots, v_s, w_1, \dots, w_t; P_1, \dots, P_m, k, \varepsilon),$$

then

$$(A_1, \dots, A_n, D) \in \Gamma^{(top)}(x_1, \dots, x_n, d : v_1, \dots, v_s, w_1, \dots, w_t; P'_1, \dots, P'_{m_1}, k, \varepsilon)$$

$$(B_1, \dots, B_p, D) \in \Gamma^{(top)}(y_1, \dots, y_p, d : v_1, \dots, v_s, w_1, \dots, w_t; P''_1, \dots, P''_{m_2}, k, \varepsilon)$$

where $P'_1, \dots, P'_{m_1} \in \mathbb{C}\langle X_1, \dots, X_{n+1}, V_1, \dots, V_s, W_1, \dots, W_t \rangle$ and $P''_1, \dots, P''_{m_2} \in \mathbb{C}\langle Y_1, \dots, Y_{p+1}, V_1, \dots, V_s, W_1, \dots, W_t \rangle$. Let $\{\mathcal{U}(A_1^\lambda, \dots, A_n^\lambda, D^\lambda); \frac{r\omega}{24}\}_{\lambda \in \Lambda_k}$ be a set of $\frac{r\omega}{24}$ -orbit-balls that cover

$$\Gamma^{(top)}(x_1, \dots, x_n, d : v_1, \dots, v_s, w_1, \dots, w_t; k, \varepsilon, P'_1, \dots, P'_{m_1})$$

with the cardinality of Λ_k satisfying

$$|\Lambda_k| = o_2 \left(\Gamma^{(top)}(x_1, \dots, x_n, d : v_1, \dots, v_s, w_1, \dots, w_t; k, \varepsilon, P'_1, \dots, P'_{m_1}); \frac{r\omega}{24} \right).$$

Also let $\{\mathcal{U}(B_1^\sigma, \dots, B_p^\sigma, D^\sigma); \frac{r\omega}{24}\}_{\sigma \in \Sigma_k}$ be a set of $\frac{r\omega}{24}$ -orbit-balls that cover

$$\Gamma_R(y_1, \dots, y_p, d : v_1, \dots, v_s, w_1, \dots, w_t; k, \varepsilon, P''_1, \dots, P''_{m_2})$$

with the cardinality of Σ_k satisfying

$$|\Sigma_k| = o_2 \left(\Gamma_R(y_1, \dots, y_p, d : v_1, \dots, v_s, w_1, \dots, w_t; k, \varepsilon, P''_1, \dots, P''_{m_2}); \frac{r\omega}{24} \right).$$

When m is sufficiently large and ε is sufficiently small, we may assume that all D^σ, D^λ are diagonal matrices in Lemma 3 by Lemma 1. For any

$$(A_1, \dots, A_n, B_1, \dots, B_p, D)$$

$$\in \Gamma_R(x_1, \dots, x_n, y_1, \dots, y_p, d : v_1, \dots, v_s, w_1, \dots, w_t; P_1, \dots, P_m, k, \varepsilon),$$

there exist some $\lambda \in \Lambda_k, \sigma \in \Sigma_k$ and $W_1, W_2 \in \mathcal{U}_k$ such that

$$\|(A_1, \dots, A_n, D) - W_1(A_1^\lambda, \dots, A_n^\lambda, D^\lambda)W_1^*\|_2 \leq \frac{r\omega}{24},$$

$$\|(B_1, \dots, B_p, D) - W_2(B_1^\sigma, \dots, B_p^\sigma, D^\sigma)W_2^*\|_2 \leq \frac{r\omega}{24}.$$

Therefore

$$\|W_1 D^\lambda W_1^* - W_2 D^\sigma W_2^*\|_2 = \|W_2^* W_1 D^\lambda - D^\sigma W_2^* W_1\|_2 \leq \frac{r\omega}{12}.$$

From Lemma 3, there exists a set $\{Ball(U_{\lambda, \sigma, \gamma}, \frac{\omega}{3})\}_{\gamma \in \Delta_k}$ in $\mathcal{U}(k)$ which cover $\Omega(D^\lambda, D^\sigma; \frac{r\omega}{12})$ with cardinality $|\Delta_k| \leq (\frac{18}{\omega})^{4rk^2}$. This implies that

$$\begin{aligned} & \|(A_1, \dots, A_n, B_1, \dots, B_p, D) \\ & - (W_2 U_{\lambda, \sigma, \gamma} A_1^\lambda U_{\lambda, \sigma, \gamma}^* W_2^*, \dots, W_2 U_{\lambda, \sigma, \gamma} A_n^\lambda U_{\lambda, \sigma, \gamma}^* W_2^*, W_2 B_1^\sigma W_2^*, \dots, W_2 B_p^\sigma W_2^*)\|_2 < n\omega \end{aligned}$$

Then we get

$$\begin{aligned} & \mathfrak{K}_3^{(top)}(x_1, \dots, x_n, y_1, \dots, y_p, u : v_1, \dots, v_s, w_1, \dots, w_t; 2n\omega) \\ & \leq \inf_{m \in \mathbb{N}, \varepsilon > 0} \limsup_{k \rightarrow \infty} \frac{\log(|\Lambda_k| |\Sigma_k| |\Delta_k|)}{k^2} \leq 4r(\log(18) - \log \omega) \end{aligned}$$

Because r is an arbitrarily small number, we have

$$\mathfrak{K}_3^{(top)}(x_1, \dots, x_n, y_1, \dots, y_p, d : v_1, \dots, v_s, w_1, \dots, w_t) = 0.$$

Note that $C^*(x_1, \dots, x_n, y_1, \dots, y_p, d) = C^*(x_1, \dots, x_n, y_1, \dots, y_p)$, so by Theorem 3, we have

$$\mathfrak{K}_3^{(top)}(x_1, \dots, x_n, y_1, \dots, y_p : v_1, \dots, v_s, w_1, \dots, w_t) = 0.$$

This completes the proof. \square

THEOREM 11. *Suppose \mathcal{A} is a C*-algebra, \mathcal{N}_1 and \mathcal{N}_2 are C*-subalgebra of \mathcal{A} . If $\mathcal{N}_1 \cap \mathcal{N}_2$ is has no finite-dimensional representation, then*

$$\mathfrak{K}_3^{(top)}(C^*(\mathcal{N}_1 \cup \mathcal{N}_2)) \leq \mathfrak{K}_3^{(top)}(\mathcal{N}_1) + \mathfrak{K}_3^{(top)}(\mathcal{N}_2)$$

PROOF. If one of $\mathfrak{K}_3^{(top)}(\mathcal{N}_1)$ and $\mathfrak{K}_3^{(top)}(\mathcal{N}_2)$ is infinity, the inequality holds automatically.

Now suppose that

$$\mathfrak{K}_3^{(top)}(\mathcal{N}_1) = \mathfrak{K}_3^{(top)}(\mathcal{N}_2) = 0.$$

By the same argument in the first part proof of Lemma 5, we can find a self-adjoint element d in $\mathcal{N}_1 \cap \mathcal{N}_2$ with $\sigma(d) = [0, 1]$. Let $\mathcal{G} = \mathcal{N}_1 \cup \mathcal{N}_2$ and $A_0 = \{d\}$. Then \mathcal{G} is a generating set of $C^*(\mathcal{N}_1 \cup \mathcal{N}_2)$. Suppose $A_0 \subseteq A \subseteq \mathcal{G}$ where $A = \{x_1, \dots, x_n, d, y_1, \dots, y_p\}$ is a finite subset with $x_1, \dots, x_n \in \mathcal{N}_1$ and $y_1, \dots, y_p \in \mathcal{N}_2$. Since $\mathfrak{K}_3^{(top)}(\mathcal{N}_1) = \mathfrak{K}_3^{(top)}(\mathcal{N}_2) = 0$, there exists

$$v_1, \dots, v_s \in \mathcal{N}_1, w_1, \dots, w_t \in \mathcal{N}_2$$

such that

$$\mathfrak{K}_2^{(top)}(x_1, \dots, x_n, d : v_1, \dots, v_s) = \mathfrak{K}_2^{(top)}(y_1, \dots, y_t, d : w_1, \dots, w_t) = 0.$$

Because $d \in C^*(x_1, \dots, x_n, d) \cap C^*(y_1, \dots, y_p, d)$, then from Lemma 5, we know that

$$\begin{aligned} & \mathfrak{K}_3^{(top)}(A : v_1, \dots, v_s, w_1, \dots, w_t) \\ &= \mathfrak{K}_3^{(top)}(x_1, \dots, x_n, y_1, \dots, y_p, d : v_1, \dots, v_s, w_1, \dots, w_t) = 0 \end{aligned}$$

Therefore, by Theorem 9 (5), $\mathfrak{K}_3^{(top)}(C^*(\mathcal{N}_1 \cup \mathcal{N}_2)) = 0$. This completes the proof. \square

THEOREM 12. *Let \mathcal{N} be an MF C^* -algebra and $\mathcal{A} \subseteq \mathcal{N}$ be a C^* -subalgebra such that \mathcal{A} has a self-adjoint element a with $\sigma(a) = [0, 1]$. If there is an unitary $u \in \mathcal{N}$ such that $u^*au \subseteq \mathcal{A}$. Then*

$$\mathfrak{K}_3^{(top)}(C^*(\mathcal{A} \cup \{u\})) \leq \mathfrak{K}_3^{(top)}(\mathcal{A})$$

PROOF. If $\mathfrak{K}_3^{(top)}(\mathcal{A}) = \infty$, we are done. Now suppose that $\mathfrak{K}_3^{(top)}(\mathcal{A}) = 0$. Let $x_1, \dots, x_n, a, u^*au$ be elements in \mathcal{A} . Then there exist y_1, \dots, y_p in \mathcal{A} such that

$$\mathfrak{K}_2^{(top)}(x_1, \dots, x_n, a, u^*au : y_1, \dots, y_p) = 0.$$

For any $0 < \omega < 1, 0 < r < 1, m, k \in \mathbb{N}$ and $\varepsilon > 0$, there exists a set

$$\left\{ \mathcal{U} \left(T_1^\lambda, \dots, T_n^\lambda, A^\lambda, B^\lambda; \frac{r\omega}{64} \right) \right\}_{\lambda \in \Lambda_k}$$

of $\frac{r\omega}{64}$ -orbit-balls in $\mathcal{M}_k(\mathbb{C})^{n+2}$ that cover

$$\Gamma^{(top)}(x_1, \dots, x_n, a, u^*au : y_1, \dots, y_p : k, \varepsilon, P_1, \dots, P_m)$$

where $P_1, \dots, P_m \in \mathbb{C}\langle X_1, \dots, X_{n+2}, Y_1, \dots, Y_p \rangle$ with the cardinality of Λ_k satisfying

$$|\Lambda_k| = o_2 \left(\Gamma^{(top)}(x_1, \dots, x_n, a, u^*au : y_1, \dots, y_p : k, \varepsilon, P_1, \dots, P_m), \frac{r\omega}{64} \right).$$

When m is sufficiently large and ε is sufficiently small, we can assume that A^λ is in the form in Lemma 3, and $B^\lambda = U^*A^\lambda U$ for some unitary matrix U .

For sufficiently large m' and sufficiently small $\varepsilon (\leq \frac{r\omega}{64})$, when

$$\begin{aligned} & (T_1, \dots, T_n, A, B, C, D) \\ & \in \Gamma^{(top)} \left(x_1, \dots, x_n, a, u^*au, \frac{u+u^*}{2}, \frac{u-u^*}{2i} : y_1, \dots, y_p; P'_1, \dots, P'_{m'}, k, \varepsilon \right). \end{aligned}$$

where $P'_1, \dots, P'_{m'} \in \mathbb{C}\langle X_1, \dots, X_{n+4}, Y_1, \dots, Y_p \rangle$. We may assume that $\|C + iD\| \leq \frac{3}{2}$ as m' large enough and ε small enough. In addition, it is clear that

$$\|A(C + iD) - (C + iD)B\| \leq \varepsilon$$

and

$$(T_1, \dots, T_n, A, B) \in \Gamma^{(top)}(x_1, \dots, x_n, a, u^*au : y_1, \dots, y_p; k, \varepsilon, P_1, \dots, P_m).$$

for some m . So there exists some $\lambda \in \Lambda_k$ and $V \in \mathcal{U}_k$ such that

$$\|(T_1, \dots, T_n, A, B) - (VT_1^\lambda V^*, \dots, VT_n^\lambda V^*, VA^\lambda V^*, VB^\lambda V^*)\|_2 \leq \frac{r\omega}{64}.$$

By Lemma 3, $Ball(U^\lambda; \frac{\omega}{2})$ of $\frac{\omega}{2}$ -balls in $\mathcal{U}(k)$ that cover $\Omega(A^\lambda, B^\lambda, \frac{r\omega}{8})$ with the cardinality of Σ_k satisfying $|\Sigma_k| \leq (\frac{12}{\omega})^{4rk^2}$. Since

$$\begin{aligned} & \|A(C + iD) - (C + iD)B\|_2 \\ & \leq \|A(C + iD) - (C + iD)B\|_2 \\ & \leq \|A(C + iD) - (C + iD)B\| \leq \varepsilon \leq \frac{r\omega}{64}, \end{aligned}$$

Then

$$\begin{aligned} & \|V^*(C + iD)VA^\lambda - B^\lambda V^*(C + iD)V\|_2 \\ & = \|(C + iD)VA^\lambda V^* - VB^\lambda V^*(C + iD)\|_2 \leq \frac{r\omega}{16}. \end{aligned}$$

Since $C + iD$ is very close to a unitary, then we may find an unitary $U^\lambda \in \Omega(A^\lambda, B^\lambda, \frac{r\omega}{8})$ such that

$$\|V^*(C + iD)V - U^\lambda\|_2 < \frac{r\omega}{4}.$$

It implies that

$$\begin{aligned} & \left\| (T_1, \dots, T_n, C, D) - \left(VT_1^\lambda V^*, \dots, VT_n^\lambda V^*, V \frac{U^\lambda + U^{\lambda*}}{2} V^*, V \frac{U^\lambda - U^{\lambda*}}{2i} V^* \right) \right\|_2 \\ & \leq r\omega \end{aligned}$$

Therefore

$$\begin{aligned} & o_2 \left(\Gamma^{(top)} \left(x_1, \dots, x_n, \frac{u + u^*}{2}, \frac{u - u^*}{2i} : a, uau^*, y_1 \dots, y_p; k, \varepsilon, P_1, \dots, P_m \right), \omega \right) \\ & \leq |\Lambda_k| |\Sigma_k| \end{aligned}$$

Hence, we get

$$\begin{aligned} 0 & \leq \mathfrak{K}_2^{(top)} \left(x_1, \dots, x_n, \frac{u + u^*}{2}, \frac{u - u^*}{2i} : a, uau^*, y_1 \dots, y_p, \omega \right) \\ & \leq \inf_{m \in \mathbb{N}, \varepsilon > 0} \limsup_{k \rightarrow \infty} \frac{\log(|\Lambda_k| |\Sigma_k|)}{k^2} \\ & \leq \inf_{m \in \mathbb{N}, \varepsilon > 0} \limsup_{k \rightarrow \infty} \left(\frac{\log(|\Lambda_k|)}{k^2} + 4r(\log 12 - \log \omega) \right) \\ & = 4r(\log 12 - \log \omega) \end{aligned}$$

Since r is an arbitrarily small positive number, we have

$$\mathfrak{K}_2^{(top)} \left(x_1, \dots, x_n, \frac{u + u^*}{2}, \frac{u - u^*}{2i} : a, uau^*, y_1 \dots, y_p, \omega \right) = 0.$$

Therefore

$$\mathfrak{K}_3^{(top)} \left(x_1, \dots, x_n, \frac{u + u^*}{2}, \frac{u - u^*}{2i} : a, uau^*, y_1 \dots, y_p \right) = 0.$$

Hence

$$\mathfrak{K}_3^{(top)}(C^*(\mathcal{A} \cup \{u\})) = 0.$$

□

THEOREM 13. *Let \mathcal{N} be an MF C^* -algebra and $\mathcal{A} \subseteq \mathcal{N}$ be a C^* -subalgebra where \mathcal{A} has no finite-dimensional representations. If there is an unitary $u \in \mathcal{N}$ such that $u\mathcal{A}u^* \subseteq \mathcal{A}$ for some u . Then*

$$\mathfrak{K}_3^{(top)}(C^*(\mathcal{A} \cup \{u\})) \leq \mathfrak{K}_3^{(top)}(\mathcal{A})$$

PROOF. If $\mathfrak{K}_3^{(top)}(\mathcal{A}) = \infty$, the inequality is clear. Now suppose that $\mathfrak{K}_3^{(top)}(\mathcal{A}) = 0$. Same as the first part proof of Lemma 5, we can find an element $a \in \mathcal{A}$ satisfying $\sigma(a) = [0, 1]$. Then the inequality holds by Theorem 12. \square

COROLLARY 4. *Suppose \mathcal{A} is a unital MF algebra with no finite-dimensional representations and \mathcal{B} is a unital MF algebra. Suppose G is a countable group of actions $\{\alpha_g\}_{g \in G}$ on \mathcal{A} . Suppose $\mathcal{D} = \mathcal{A} \rtimes_\alpha G$ is either a full or reduced crossed product of \mathcal{A} by the actions of G . If there is a onto $*$ -homomorphism $\pi : \mathcal{A} \rtimes_\alpha G \rightarrow \mathcal{B}$, then*

$$\mathfrak{K}_3^{(top)}(\mathcal{B}) \leq \mathfrak{K}_3^{(top)}(\mathcal{A}).$$

PROOF. Since \mathcal{A} has no finite-dimensional representations, then $\pi(\mathcal{A})$ has no finite-dimensional representations. Therefore, we can find an element $\pi(a) \in \pi(\mathcal{A})$ with $\sigma(\pi(a)) = [0, 1]$. Note that $\pi(g^{-1})\pi(a)\pi(g) \subseteq \pi(\mathcal{A})$, then by Theorem 13, $\mathfrak{K}_3^{(top)}(\pi(\mathcal{A}) \cup \{\pi(g)\}) = 0$. From Theorem 11, we know that

$$\mathfrak{K}_3^{(top)}(\pi(\mathcal{A}) \cup \{\pi(g_1)\} \cup \{\pi(g_2)\}) = 0.$$

Let

$$\mathcal{B}_n = C^*(\pi(\mathcal{A}) \cup \{\pi(g_1)\} \cup \dots \cup \{\pi(g_n)\}).$$

Then $\mathfrak{K}_3^{(top)}(\mathcal{B}_n) = 0$.

Therefore

$$\mathfrak{K}_3^{(top)}(\mathcal{B}) = \liminf_n \mathfrak{K}_3^{(top)}(\mathcal{B}_n) = 0$$

by the fact that $\mathcal{B} = \overline{\bigcup_n \mathcal{B}_n}^{\|\cdot\|}$ and Corollary 3. \square

6. Applications to Central Sequence Algebras

In [19] Weihua Li and Junhao Shen proved that a separable approximately divisible C^* -algebra \mathcal{A} is singly generated and that if $\mathcal{A} = C^*(x_1, \dots, x_n)$ is an MF-algebra, then

$$\delta^{\text{top}}(x_1, \dots, x_n) = 1.$$

Later Li and Shen [20] proved that if \mathcal{A} is an approximately divisible unital C^* -algebra, then Pisier's similarity degree $d(\mathcal{A})$ is at most 5. After that Don Hadwin and Weihua Li [13] defined the larger class of weakly approximately divisible C^* -algebras and proved for these algebras the similarity degree is at most 5. More recently, Wenhua Qian and Junhao Shen [21] defined the still larger class of C^* -algebras with property $c^*\text{-}\Gamma$ and proved that the similarity degree is at most 3.

In this section we want to view these results in terms of the central sequence algebra of a separable unital C^* -algebra \mathcal{A} . Let $c_\omega(\mathcal{A})$ denote the closed two-sided ideal of the C^* -algebra $l^\infty(\mathcal{A})$ given by

$$c_\omega(\mathcal{A}) = \left\{ (a_n)_{n \geq 1} \in l^\infty(\mathcal{A}) \mid \lim_{n \rightarrow \omega} \|a_n\| = 0 \right\}.$$

The ultrapower \mathcal{A}_ω is defined to be the quotient C^* -algebra $l^\infty(\mathcal{A})/c_\omega(\mathcal{A})$, and we denote by π_ω the quotient mapping $l^\infty(\mathcal{A}) \rightarrow \mathcal{A}_\omega$. Let $l : \mathcal{A} \rightarrow l^\infty(\mathcal{A})$

denote the "diagonal" inclusion mapping $l(a) = (a, a, \dots) \in l^\infty(\mathcal{A})$, $a \in \mathcal{A}$; and put $l_\omega = \pi_\omega \circ l : \mathcal{A} \longrightarrow \mathcal{A}_\omega$. Both mapping l and l_ω are injective. If we view \mathcal{A} as a subalgebra of \mathcal{A}_ω , then the relative commutant is defined by $\mathcal{A}_\omega \cap \mathcal{A}'$ which is called a central sequence algebra of \mathcal{A} . Suppose τ is a tracial state on \mathcal{A} and \mathcal{N} is the weak closure of \mathcal{A} in the GNS representation determined by τ . The algebra $\mathcal{N}^\omega := l^\infty(\mathcal{N})/c_{\tau,\omega}(\mathcal{N})$ (with $c_{\tau,\omega}(\mathcal{N})$ the bounded sequences in \mathcal{N} with $\lim_\omega \|a\|_{2,\tau} = 0$) is a W*-algebra when ω is a free ultrafilter.

If \mathcal{M} is a II_1 factor, then \mathcal{M} has *property* Γ if and only if $\mathcal{M}^\omega \cap \mathcal{M}'$ has a representing sequence (U_1, U_2, \dots) such that each U_n is a Haar unitary element of \mathcal{M} (i.e., $\tau(U^n) = 0$ for all $n \in \mathbb{N}$). If \mathcal{M} is a II_1 von Neumann algebra with a separable predual, then \mathcal{M} is defined in [21] to have *property* Γ if and only if each II_1 factor in the central decomposition of \mathcal{M} has property Γ . It follows from direct integral theory that if \mathcal{M} has property Γ , then $\mathcal{M}^\omega \cap \mathcal{M}'$ contains a representing sequence of Haar unitaries. The following Theorem duo to Dixmier [7] and Connes [5].

THEOREM 14. *Let \mathcal{M} be a separable II_1 factor. The following conditions are equivalent:*

- (1) \mathcal{M} has *property* Γ ;
- (2) $\mathcal{M}^\omega \cap \mathcal{M}' \neq \mathbb{C}I$;
- (3) $\mathcal{M}^\omega \cap \mathcal{M}'$ is a *diffuse von Neumann algebra*.

let $\pi_\tau(\mathcal{A})''$ be the weak closure of \mathcal{A} under the GNS representation of \mathcal{A} with respect to the state τ .

In [21], a separable unital C*-algebra is said to have *property* $c^*\text{-}\Gamma$ if, for every tracial state τ on \mathcal{A} such that $\pi_\tau(\mathcal{A})''$ is a II_1 factor, $\pi_\tau(\mathcal{A})''$ has property Γ , which is equivalent to $\pi_\tau(\mathcal{A})''$ having property Γ whenever $\pi_\tau(\mathcal{A})''$ is a II_1 von Neumann algebra. If \mathcal{A} has no finite-dimensional representations, then so does $\pi_\tau(\mathcal{A})''$. Therefore $\pi_\tau(\mathcal{A})''$ is II_1 for every tracial state τ on \mathcal{A} . So if \mathcal{A} has no finite-dimensional representations and property $c^*\text{-}\Gamma$, then $\pi_\tau(\mathcal{A})''$ has property Γ for every tracial state τ on \mathcal{A} . Actually we can say more in this case.

LEMMA 6. *Suppose \mathcal{A} is a separable unital C*-algebra having no finite-dimensional representations. Then \mathcal{A} has property $c^*\text{-}\Gamma$ if and only if for every tracial state τ on \mathcal{A} , the central sequence algebra of $\pi_\tau(\mathcal{A})''$ has no finite-dimensional representations.*

PROOF. If \mathcal{A} has property $c^*\text{-}\Gamma$, then $\pi_\tau(\mathcal{A})''$ has property Γ for every tracial state τ on \mathcal{A} . So the central sequence algebra of each II_1 factor in the central decomposition of $\pi_\tau(\mathcal{A})''$ has no finite-dimensional representations by Theorem 14.

On the other hand, if the central sequence algebra of $\pi_\tau(\mathcal{A})''$ has no finite-dimensional representations for every tracial state, then the central sequence algebra of each II_1 factor in the central decomposition of $\pi_\tau(\mathcal{A})''$ has no finite-dimensional representations. So by Theorem 14, \mathcal{A} has property $c^*\text{-}\Gamma$ \square

REMARK 11. *Let \mathcal{A} be a separable unital C*-algebra, let τ be a tracial state on \mathcal{A} , let \mathcal{N} be the weak closure of \mathcal{A} under the GNS representation of \mathcal{A} with respect to the state τ , and let ω be a free ultrafilter on \mathbb{N} . It follows that there are the following two natural *-homomorphisms*

$$\mathcal{A}_\omega \longrightarrow \mathcal{N}^\omega, \quad \mathcal{A}_\omega \cap \mathcal{A}' \longrightarrow \mathcal{N}^\omega \cap \mathcal{N}'.$$

LEMMA 7. ([23]) Let \mathcal{A} be a separable unital C^* -algebra, let τ be a faithful tracial state on \mathcal{A} , let \mathcal{N} be the weak closure of \mathcal{A} under the GNS representation of \mathcal{A} with respect to the state τ , and let ω be a free ultrafilter on \mathbb{N} . It follows that the natural $*$ -homomorphisms

$$\mathcal{A}_\omega \longrightarrow \mathcal{N}^\omega, \quad \mathcal{A}_\omega \cap \mathcal{A}' \longrightarrow \mathcal{N}^\omega \cap \mathcal{N}'$$

are surjective.

We say that an MF algebra \mathcal{A} with no finite-dimensional representations has property MF- c^* - Γ if, for every MF-trace τ on \mathcal{A} , the central sequence algebra $(\pi_\tau(\mathcal{A})'')^\omega \cap \pi_\tau(\mathcal{A})'$ has no finite-dimensional representations, i.e., $\pi_\tau(\mathcal{A})''$ has property Γ .

THEOREM 15. ([12]) If \mathcal{M} is a von Neumann algebra with a central net of Haar unitaries, then $\mathfrak{K}_3(\mathcal{M}) = 0$.

THEOREM 16. Let \mathcal{A} be a unital MF C^* -algebra with no finite-dimensional representations. If \mathcal{A} has property c^* - Γ , then $\mathfrak{K}_3^{(top)}(\mathcal{A}) = 0$.

PROOF. Let $\mathcal{N} = \pi_\tau(\mathcal{A})''$ be the weak closure of \mathcal{A} under the GNS representation of \mathcal{A} with respect to the tracial state τ . Since \mathcal{A} has property c^* - Γ , then there is a central sequence $\{u_n\}$ of Haar unitaries in \mathcal{N} such that $[\{u_n\}] = u \in \mathcal{N}^\omega \cap \mathcal{N}'$. It follows that $\mathfrak{K}_3(\mathcal{A}; \tau) = 0$ by Theorem 15. Hence $\mathfrak{K}_3^{(3)}(\mathcal{A}; \tau) = 0$ by Remark 9. Since

$$\mathfrak{K}_{top}^{(3)}(\mathcal{A}) \leq \sup_{\tau \in TS(\mathcal{A})} \mathfrak{K}_3^{(3)}(\mathcal{A}; \tau) = 0,$$

by Theorem 10, then $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$. \square

COROLLARY 5. Let \mathcal{A} be an MF algebra with no finite-dimensional representations. Suppose each tracial state on \mathcal{A} is faithful. If $\mathcal{A}_\omega \cap \mathcal{A}'$ has no finite-dimensional representations, then $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$.

PROOF. let \mathcal{N} be the weak closure of \mathcal{A} under the GNS representation of \mathcal{A} with respect to the tracial state τ . Since τ is faithful, the natural $*$ -homomorphisms

$$\mathcal{A}_\omega \cap \mathcal{A}' \longrightarrow \mathcal{N}^\omega \cap \mathcal{N}'$$

is surjective by Lemma 7. It follows that $\mathcal{N}^\omega \cap \mathcal{N}'$ has no finite-dimensional representation, hence \mathcal{A} has property c^* - Γ by Lemma 6. Therefore $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$ by Theorem 16. \square

COROLLARY 6. Let $\mathcal{A} = C^*(x_1, x_2, \dots, x_n)$ be a unital MF C^* -algebra with no finite-dimensional representations. If \mathcal{A} has property MF- c^* - Γ , then $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$.

PROOF. Let $\mathcal{N} = \pi_\tau(\mathcal{A})''$ be the weak closure of \mathcal{A} under the GNS representation of \mathcal{A} with respect to the tracial state τ . Since \mathcal{A} has property MF- c^* - Γ , $\mathcal{N}^\omega \cap \mathcal{N}'$ has no finite-dimensional representation. Then there is a central sequence $\{u_n\}$ of Haar unitaries in \mathcal{N} such that $[\{u_n\}] = u \in \mathcal{N}^\omega \cap \mathcal{N}'$. It follows that $\mathfrak{K}_3(\mathcal{A}; \tau) = 0$ by Theorem 15. Hence $\mathfrak{K}_3^{(3)}(\mathcal{A}; \tau) = 0$ by Remark 9. It implies that $\mathfrak{K}_2^{(2)}(\mathcal{A}; \tau) = 0$ for every MF tracial state τ . Note

$$\mathfrak{K}_{top}^{(2)}(\mathcal{A}) \leq \sup_{\tau \in \mathcal{T}_{MF}(\mathcal{A})} \mathfrak{K}_2^{(2)}(\mathcal{A}; \tau) = 0,$$

by Theorem 5, then $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$ by Remark 8. \square

REMARK 12. *We don't know whether property MF-c*- Γ is equivalent to $\mathfrak{K}_{top}^{(3)}(\mathcal{A}) = 0$ in which $\mathcal{A} = C^*(x_1, x_2, \dots, x_n)$ has no finite-dimensional representations. But it is well-known that $C_r^*(\mathbb{F}_2)$ is simple, hence $C_r^*(\mathbb{F}_2)$ has no finite-dimensional representation. And it is obvious that $C_r^*(\mathbb{F}_2)$ has no property MF-c*- Γ , so we may hope $\mathfrak{K}_{top}^{(3)}(C_r^*(\mathbb{F}_2)) = \infty$, i.e., $\mathfrak{K}_{top}^{(2)}(C_r^*(\mathbb{F}_2)) \neq 0$. Actually, Voiculescu [24] proved that $\delta_{top}(S_1, S_2) = 2$, where S_1 and S_2 are free semicircle elements. Therefore $\mathfrak{K}_{top}^{(2)}(C_r^*(\mathbb{F}_2)) \neq 0$ by Theorem 3.1.2 in [11].*

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